

AUTOMORPHISMS OF \overline{T}

INDRANIL BISWAS, S. SENTHAMARAI KANNAN, AND D. S. NAGARAJ

ABSTRACT. Let \overline{G} be the wonderful compactification of a simple affine algebraic group G defined over \mathbb{C} such that its center is trivial and $G \neq \mathrm{PSL}(2, \mathbb{C})$. Take a maximal torus $T \subset G$, and denote by \overline{T} its closure in \overline{G} . We prove that T coincides with the connected component, containing the identity element, of the group of automorphisms of the variety \overline{T} .

Résumé. *Automorphismes de \overline{T}*

Soit \overline{G} la compactification magnifique d'un groupe algébrique affine G défini sur \mathbb{C} , dont le centre est trivial et tel que $G \neq \mathrm{PSL}(2, \mathbb{C})$. Soit $T \subset G$ un tore maximal, et soit \overline{T} son adhérence dans \overline{G} . Nous montrons que T est égal à la composante connexe contenant l'élément neutre du groupe d'automorphismes de la variété \overline{T} .

1. INTRODUCTION

Let G be a simple affine algebraic group defined over the complex numbers such that the center of G is trivial. De Concini and Procesi constructed a very interesting compactification of G which is known as the wonderful compactification [DP, p. 14, 3.1, THEOREM]. The wonderful compactification of G will be denoted by \overline{G} . Fix a maximal torus T of G . Let \overline{T} denote the closure of T in \overline{G} . The connected component, containing the identity element, of the group of all automorphisms of the variety \overline{T} will be denoted by $\mathrm{Aut}^0(\overline{T})$. For more details about the variety \overline{T} we refer to [BJ, § 1]. Our aim here is to compute $\mathrm{Aut}^0(\overline{T})$.

Using the action of G on \overline{G} , we have $T \subset \mathrm{Aut}^0(\overline{T})$; this inclusion does not depend on whether the right or the left action is chosen. We prove that $T = \mathrm{Aut}^0(\overline{T})$, provided $G \neq \mathrm{PSL}(2, \mathbb{C})$; see Theorem 3.1.

Note that $\mathrm{Aut}(\overline{T})$ is not connected since \overline{T} is stable under the conjugation of the normalizer $N_G(T)$ of T in G .

If $G = \mathrm{PSL}(2, \mathbb{C})$, then $\overline{T} = \mathbb{P}^1$, and hence $\mathrm{Aut}^0(\overline{T}) = \mathrm{PSL}(2, \mathbb{C})$.

2. LIE ALGEBRA AND ALGEBRAIC GROUPS

In this section we recall some basic facts and notation on Lie algebra and algebraic groups (see [Hu1], [Hu2] for details). Throughout G denotes an affine algebraic group over \mathbb{C} which is simple and of adjoint type. We also assume that the rank of G is at-least two, equivalently $G \neq \mathrm{PSL}(2, \mathbb{C})$.

2010 *Mathematics Subject Classification.* 14L10, 14L30.

Key words and phrases. Wonderful compactification, closure of T , automorphism group.

For a maximal torus T of G , the group of all characters of T will be denoted by $X(T)$. The Weyl group of G with respect to T is defined to be $W := N_G(T)/T$, where $N_G(T)$ is the normalizer of T in G . By $R \subset X(T)$ we denote the root system of G with respect to T . For a Borel subgroup B of G containing T , let $R^+(B)$ denote the set of positive roots determined by T and B . Let

$$S = \{\alpha_1, \dots, \alpha_n\}$$

be the set of simple roots in $R^+(B)$. Let B^- denote the opposite Borel subgroup of G determined by B and T . For $\alpha \in R^+(B)$, let $s_\alpha \in W$ be the reflection corresponding to α . The Lie algebras of G , T and B will be denoted by \mathfrak{g} , \mathfrak{t} and \mathfrak{b} respectively. The dual of the real form $\mathfrak{t}_{\mathbb{R}}$ of \mathfrak{t} is $X(T) \otimes \mathbb{R} = \text{Hom}_{\mathbb{R}}(\mathfrak{t}_{\mathbb{R}}, \mathbb{R})$.

The positive definite W -invariant form on $\text{Hom}_{\mathbb{R}}(\mathfrak{t}_{\mathbb{R}}, \mathbb{R})$ induced by the Killing form on \mathfrak{g} is denoted by $(\ , \)$. We use the notation

$$\langle \nu, \alpha \rangle := \frac{2(\nu, \alpha)}{(\alpha, \alpha)}.$$

In this setting one has the Chevalley basis

$$\{x_\alpha, h_\beta \mid \alpha \in R, \beta \in S\}$$

of \mathfrak{g} determined by T . For a root α , we denote by U_α (respectively, \mathfrak{g}_α) the one-dimensional T stable root subgroup of G (respectively, the subspace of \mathfrak{g}) on which T acts through the character α .

Now, let σ be the involution of $G \times G$ defined by $\sigma(x, y) = (y, x)$. Note that the diagonal subgroup $\Delta(G)$ of $G \times G$ is the subgroup of fixed points, while $T \times T$ is a σ -stable maximal torus of $G \times G$ and $B \times B^-$ is a Borel subgroup having the property that $\sigma(\alpha) \in -R^+(B \times B^-)$ for every $\alpha \in R^+(B \times B^-)$.

Let \overline{G} denote the wonderful compactification of the group G , where G is identified with the symmetric space $(G \times G)/\Delta(G)$ (see [DP, p. 14, 3.1. THEOREM]). Let \overline{T} be the closure of T in \overline{G} .

3. THE CONNECTED COMPONENT OF THE AUTOMORPHISM GROUP

Recall that if X is a smooth projective variety over \mathbb{C} , the connected component of the group of all automorphisms of X containing the identity automorphism is an algebraic group (see [MO, p. 17, Theorem 3.7] and [Gr, p. 268]), which deal also the case when X may be singular or it may be defined over any field). Further, the Lie algebra of this automorphism group is isomorphic to the space of all vector fields on X , that is the space $H^0(X, \Theta_X)$ of all global sections of the tangent bundle Θ_X of X (see [MO, p. 13, Lemma 3.4]).

Let $\text{Aut}(\overline{T})$ denote the group of all algebraic automorphisms of the variety \overline{T} . Let

$$\text{Aut}^0(\overline{T}) \subset \text{Aut}(\overline{T})$$

be the connected component containing the identity element. We note that $\text{Aut}^0(\overline{T})$ is an algebraic group with Lie algebra $H^0(\overline{T}, \Theta_{\overline{T}})$, where $\Theta_{\overline{T}}$ is the tangent bundle of the variety \overline{T} ; the Lie algebra structure on $H^0(\overline{T}, \Theta_{\overline{T}})$ is given by the Lie bracket of vector fields.

The subvariety $\overline{T} \subset \overline{G}$ is stable under the action of $T \times T$. Further, the subgroup $T \times 1 \subset T \times T$ acts faithfully on \overline{T} , and $T \subset \overline{T}$ is a stable Zariski open dense subset for this action of T . Hence, we get an injective homomorphism

$$\rho : T \longrightarrow \text{Aut}^0(\overline{T}).$$

Theorem 3.1. *The above homomorphism ρ is an isomorphism.*

Proof. We know that T is a maximal torus of $\text{Aut}^0(\overline{T})$ [De, p. 521, COROLLAIRE 1]. Choose a Borel subgroup $B' \subset \text{Aut}^0(\overline{T})$ containing the maximal torus T of $\text{Aut}^0(\overline{T})$. The action of B' on \overline{T} fixes a point in because \overline{T} is a projective variety (see [Hu2, p. 134, 21.2, Theorem]). Let $x \in \overline{T}$ be a point fixed by B' . Clearly, $n\overline{T}n^{-1} = \overline{T}$ for $n \in N_G(T)$, and the diagonal subgroup of $T \times T$ acts trivially on \overline{T} . Hence $W = N_G(T)/T$ is a subgroup of $\text{Aut}(\overline{T})$. The diagonal subgroup of $T \times T$ acts trivially on \overline{T} . So we see that $T \times T$ fixes the point x . Therefore, by [BJ, p. 477, (1.2.7)] and [BJ, p. 478, (1.3.8)] we have that $x = w(z)$ for some $w \in W$, where z is the unique $B \times B^-$ fixed point in \overline{G} . Using conjugation by w^{-1} , we may assume that B' fixes z . Let

$$Q \subset \text{Aut}^0(\overline{T})$$

be the stabilizer subgroup for the point z . As $B' \subset Q$, it follows that Q is in fact a parabolic subgroup of $\text{Aut}^0(\overline{T})$.

We first show that $\text{Aut}^0(\overline{T})$ is reductive. Let R_u be the unipotent radical of $\text{Aut}^0(\overline{T})$. Therefore, R_u is also the unipotent radical of $\text{Aut}(\overline{T})$. Hence $wR_uw^{-1} = R_u$ for all $w \in W$. Consequently, $R_u \subset B'$ fixes $w(z)$ for every $w \in W$.

For $\chi \in X(B) = X(T)$, let \mathcal{L}_χ be the line bundle on \overline{G} associated to χ (see [DP, p. 26, 8.1, PROPOSITION]). Take any $w \in W$. The action of R_u fixes $w(z)$, so the fiber $(\mathcal{L}_\chi)_{w(z)}$ of \mathcal{L}_χ over $w(z)$ is an one dimensional representation of R_u . This R_u -module $(\mathcal{L}_\chi)_{w(z)}$ is trivial because the group R_u is unipotent.

Let $\mathbb{C}[T]$ be the coordinate ring of the affine algebraic group T . We note that $\mathbb{C}[T]$ is a unique factorization domain, and therefore any line bundle on T is trivial. As $T \subset \overline{T}$ is a T stable open dense subset for the left translation action, we see that the T module $H^0(\overline{T}, \mathcal{L}_\chi)$ is a submodule of $\mathbb{C}[T]$. If χ is a dominant character of T , and $w \in W$, then the weight space $\mathbb{C}[T]$ of weight $-w(\chi)$ is one dimensional and spanned by $t^{-w(\chi)}$. Moreover, we have $t^{-w(\chi)} \in H^0(\overline{T}, \mathcal{L}_\chi)$, because it is the unique section of weight $-w(\chi)$ not vanishing at $w(z)$. Thus, from the above it follows that $t^{-w(\chi)}$ is fixed by R_u for every dominant character χ of T and every $w \in W$.

The set $\{t^\chi \mid \chi \in X(T)\}$ is a basis for the complex vector space $\mathbb{C}[T]$. Therefore, the action of R_u on $H^0(\overline{T}, \mathcal{L}_\chi)$ is trivial for every regular dominant character χ of T . We have

$$\overline{T} \subset \mathbb{P}(H^0(\overline{T}, \mathcal{L}_\chi)),$$

and hence it follows that the action of R_u on \overline{T} is trivial, implying that R_u is trivial. Thus, the group $\text{Aut}^0(\overline{T})$ is reductive.

Next we will show that $Q = B'$. Fix a dominant character χ of $T \subset B$. As $\text{Aut}^0(\overline{T})$ is reductive $\text{Aut}^0(\overline{T})/Z$ is semisimple, where $Z(\subset Q)$ is the center of $\text{Aut}^0(\overline{T})$. Note that Q/Z is a parabolic subgroup of $\text{Aut}^0(\overline{T})/Z$ and it fixes z , by the arguments in the proofs of [KKV, p. 81–82, 3.2, Proposition and 3.3, Corollary] there is a positive integer a such that Q/Z acts linearly on the fiber of the line bundle $\mathcal{L}_\chi^{\otimes a}$ over z through some character χ' of Q/Z . Pulling back χ' to Q we see that Q acts on the fiber the line bundle $\mathcal{L}_\chi^{\otimes a}$ over z by a character. The group $X(T)$ is finitely generated and Abelian, and hence the image of the restriction map

$$X(Q) \longrightarrow X(B') = X(T)$$

is of finite index. This implies that the rank of $X(Q)$ is equal to the the rank of $X(B')$. Thus, we have $Q = B'$.

We will now show that $\text{Aut}^0(\overline{T})$ is not semisimple. If $\text{Aut}^0(\overline{T})$ is semisimple, then

$$\dim \overline{T} = \dim T \leq \dim B'_u = \dim(\text{Aut}^0(\overline{T})/B'), \quad (1)$$

where B'_u is the unipotent radical of B' . Note that by the above observation, B' is the stabilizer of z in $\text{Aut}^0(\overline{T})$. Since B' is a Borel subgroup of $\text{Aut}^0(\overline{T})$, $\text{Aut}^0(\overline{T})/B'$ is a closed subvariety of \overline{T} . Thus from (1) we get that $\text{Aut}^0(\overline{T})/B' = \overline{T}$. This implies that $\overline{T} = (\mathbb{P}^1)^n$, and $\text{Lie}(\text{Aut}^0(\overline{T})) = \mathfrak{sl}(2, \mathbb{C})^n$, where $n = \dim T$. The T -fixed points of $(\mathbb{P}^1)^n$ are indexed by the elements of the Weyl group of $\text{PSL}(2, \mathbb{C})^n$. Therefore, \overline{T} has 2^n fixed points for the action of T . On the other hand, by [BJ, p. 477, (1.2.7) and p. 478, (1.3.8)], all $w(z) \in \overline{T}$, $w \in W$, are fixed by T , and $w'(z) = w(z)$ only if $w' = w$. Consequently, the order of W is at most 2^n . As $n = \dim T$, this is possible only if $W = S_2^n$. Hence it follows that $G = \text{PSL}(2, \mathbb{C})^n$. But this contradicts the assumption that G is simple of rank $n \geq 2$. So $\text{Aut}^0(\overline{T})$ is not semisimple.

The group $\text{Aut}^0(\overline{T})$ is reductive but not semisimple, and this implies that the connected component Z^0 , containing the identity element, of the center of $\text{Aut}^0(\overline{T})$ is a positive dimensional sub-torus of T . Further, since

$$w\text{Aut}^0(\overline{T})w^{-1} = \text{Aut}^0(\overline{T}),$$

it follows that $wZ^0w^{-1} = Z^0$ for every $w \in W$. Thus, the restriction map

$$r : X(T) \otimes_{\mathbb{Z}} \mathbb{R} \longrightarrow X(Z^0) \otimes_{\mathbb{Z}} \mathbb{R} \quad (2)$$

is a nonzero homomorphism of W modules. Note that $X(T) \otimes \mathbb{R}$ is an irreducible W module (this is because G is simple). So we conclude that the homomorphism r in (2) is an isomorphism. Consequently, we have $T = Z^0$ and $T = \text{Aut}^0(\overline{T})$. \square

ACKNOWLEDGEMENTS

We are grateful to the referee for helpful comments. The first-named author thanks the Institute of Mathematical Sciences for hospitality while this work was carried out. He also acknowledges the support of the J. C. Bose Fellowship. The second named author would like to thank the Infosys Foundation for the partial support.

REFERENCES

- [BJ] M. Brion and R. Joshua, Equivariant Chow ring and Chern classes of wonderful symmetric varieties of minimal rank, *Transform. Groups* **13** (2008), 471–493.
- [DP] C. De Concini and C. Procesi, Complete symmetric varieties, *Invariant theory* (Montecatini, 1982), 1–44, Lecture Notes in Math., 996, Springer, Berlin, 1983.
- [De] M. Demazure, Sous-groupes algébriques de rang maximum du groupe de Cremona, *Annal. Sci. École Norm. Sup.* **3** (1970), 507–588.
- [Gr] A. Grothendieck, *Techniques de construction et théorèmes d'existence en géométrie algébrique IV, les schémas de Hilbert*, Séminaire Bourbaki **5** (1960-1961), Exposé no. 221, 28 pp.
- [Hu1] J. E. Humphreys, *Introduction to Lie algebras and Representation theory*, Springer-Verlag, Berlin Heidelberg, New York, 1972.
- [Hu2] J. E. Humphreys, *Linear Algebraic Groups*, GTM 21, Springer-Verlag, Berlin Heidelberg, New York, 1975.
- [KKV] F. Knop, H. Kraft and T. Vust, *The Picard group of a G-variety*, Algebraische Transformationssgruppen und Invariantentheorie, 7787, DMV Sem., 13, Birkhuser, Basel, 1989. 14C22 (14D25 14L30)
- [MO] H. Matsumura and F. Oort, *Representability of group functors, and automorphisms of algebraic schemes*, Invent. Math. **4** (1967), 1-25.

SCHOOL OF MATHEMATICS, TATA INSTITUTE OF FUNDAMENTAL RESEARCH, HOMI BHABHA ROAD, MUMBAI 400005, INDIA

E-mail address: `indranil@math.tifr.res.in`

CHENNAI MATHEMATICAL INSTITUTE, H1, SIPCOT IT PARK, SIRUSERI, KELAMBAKKAM 603103, INDIA

E-mail address: `kannan@cmi.ac.in`

THE INSTITUTE OF MATHEMATICAL SCIENCES, CIT CAMPUS, TARAMANI, CHENNAI 600113, INDIA

E-mail address: `dsn@imsc.res.in`